

On cohomology of saturated fusion systems and support varieties

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Abstract

In this short note we study the cohomology algebra of saturated fusion systems using finite groups which realize saturated fusion systems and Hochschild cohomology of group algebras. A similar result to a theorem of Alperin, [1] is proved for varieties of cohomology algebras of fusions systems associated to block algebras of finite groups.

Keywords: fusion system, cohomology, group algebra, variety, block

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1. Introduction

A saturated fusion system \mathcal{F} on a finite p -group P is a category whose objects are the subgroups of P and whose morphisms satisfy certain axioms mimicking the behavior of a finite group G having P as a Sylow subgroup. The axioms of saturated fusion systems were invented by Puig in early 1990's. The cohomology algebra of a p -local finite group with coefficients in \mathbb{F}_p is introduced in [4, §5]. Let k be an algebraically closed field of characteristic p . We denote by $\mathrm{HH}^*(kG)$ the Hochschild cohomology algebra of the group algebra kG and by $\mathrm{H}^*(G, k)$ the cohomology algebra of the group G with trivial coefficients. As in [8] we will use the language of homotopy classes of chain maps (see [8, 2.8, 4.2]). We denote by $\mathrm{H}^*(\mathcal{F})$ the algebra of stable elements of \mathcal{F} , i.e. the cohomology algebra of the saturated fusion system \mathcal{F} , which is the subalgebra of $\mathrm{H}^*(P, k)$ consisting of elements $[\zeta] \in \mathrm{H}^*(P, k)$ such that

$$\mathrm{res}_Q^P([\zeta]) = \mathrm{res}_\varphi([\zeta]),$$

for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ and any subgroup Q of P . This is the main object of study in this paper. Moreover Broto, Levi and Oliver showed that any saturated fusion system \mathcal{F} has

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a non-unique $P - P$ -biset X with certain properties formulated by Linckelmann and Webb (see [4, Proposition 5.5]). Such a $P - P$ -biset X is called a characteristic biset. Using this biset, S. Park noticed in [11] a result which says that a saturated fusion system can be realized by a finite group. This finite group is $G = \text{Aut}(X_P)$, that is the group of bijections of the characteristic biset X , preserving the right P -action. So, by [11, Theorem 3], we can identify \mathcal{F} with $\mathcal{F}_P(G)$ which is the fusion system on P such that for every $Q, R \leq P$ we have

$$\text{Hom}_{\mathcal{F}_P(G)}(Q, R) = \{\varphi : Q \rightarrow R \mid \exists x \in G \text{ s.t. } \varphi(u) = xux^{-1}, \forall u \in Q\}.$$

In Section 2 for a finite group G and P a p -subgroup such that $\mathcal{F}_P(G)$ is a saturated fusion system we associate and analyze a restriction map from the cohomology algebra of the group G with coefficients in the field k to the cohomology algebra of the fusion system, $H^*(\mathcal{F}_P(G))$. For shortness we denote $\mathcal{F} = \mathcal{F}_P(G)$ and this map by $\rho_{\mathcal{F}, G}$. From the above notice that any saturated fusion system \mathcal{F} can be identified with a saturated fusion system of the form $\mathcal{F}_P(G)$, where $G = \text{Aut}(X_P)$ for X a characteristic biset; but we prefer to work under a more general setup. Although some results are straightforward translations of results in the literature, to prove them we need the machinery of transfer maps between Hochschild cohomology algebras of group algebras developed in [8]. Since $H^*(\mathcal{F})$ is a graded commutative finitely generated k -algebra we can associate the spectrum of maximal ideals, i.e. the variety denoted $V_{\mathcal{F}}$. For U a finitely generated kP -module we define also a support variety, denoted $V_{\mathcal{F}}(U)$, in a similar way as the usual support variety in group cohomology. The main result of this section is Theorem 2.1, where we prove that $\rho_{\mathcal{F}, G}$ induces a finite map on varieties and that we can recover $V_{\mathcal{F}}(U)$ from the support variety $V_G(U)$ associated to any finitely generated kG -module U , which was first introduced by Carlson in [6].

In Section 3 we consider only fusion systems associated to blocks. If b is a block idempotent of the group algebra kG (i.e. a primitive idempotent in the center of the group algebra) with defect group P then it is well known that the set of b -Brauer pairs form a G -poset which defines a saturated fusion system. We denote this category by $\mathcal{F}_{G, b}$. By considering a subgroup H in G and a block c of kH with defect group $Q \leq P$ such that $\mathcal{F}_{H, c}$ is a fusion subsystem of $\mathcal{F}_{G, b}$ we can define a restriction map from the cohomology of $\mathcal{F}_{G, b}$ to the cohomology of $\mathcal{F}_{H, c}$, denoted $\text{res}_{b, c}$. In the main theorem of this short note we find necessary and sufficient conditions for this restriction to induce a bijective map from $V_{\mathcal{F}_{G, b}}$ to $V_{\mathcal{F}_{H, c}}$.

For definitions, notations and basic results from block theory and fusion system theory we follow [12] and [2].

2. Cohomology of a finite group realizing a saturated fusion system, restrictions and varieties

Let G be a finite group and P be a p -subgroup of G . We denote by M the k -module kG regarded as $kG - kP$ -bimodule. In [8] M. Linckelmann defines transfer maps between Hochschild cohomology algebras of symmetric algebras and establishes the compatibility of restriction and transfer maps for group cohomology with these transfer maps for group algebras through some embedding, which we call diagonal induction. Recall the notations of these well-known injective homomorphisms of algebras

$$\delta_G : H^*(G, k) \rightarrow HH^*(kG), \quad \delta_P : H^*(P, k) \rightarrow HH^*(kP).$$

Moreover from [8, Proposition 4.8] we have $\text{Im} \delta_G \subseteq HH_M^*(kG)$, where $HH_M^*(kG)$ is the subalgebra of M -stable elements in $HH^*(kG)$. See [8, Definition 3.1] for the general case and for the definition of projective elements. By [8, Example 3.9] we know that the projective elements are in this case $\pi_{M^*} = 1_{kP}$, which is invertible in $Z(kP)$ and $\pi_M = [G : P] 1_{kG}$. It follows that there is a normalized transfer map T_{M^*} which is equal to t_{M^*} . Let U be a finitely generated kG -module, where G is a finite group. The cohomological variety of U (denoted $V_G(U)$), introduced by Carlson in [6], is defined to be the subvariety of the maximal ideal spectrum of $H^*(G, k)$ (denoted V_G) determined by $I_G^*(U)$, that is the variety of the quotient algebra $H^*(G, k)/I_G^*(U)$. Here $I_G^*(U)$ is the kernel of the graded k -algebra homomorphism $H^*(G, k) \rightarrow \text{Ext}_{kG}^*(U, U)$ induced by the functor $- \otimes_k U$. For more details see [3] or [7]. Moreover, M. Linckelmann noticed in [9, 2.9] that the ideal $I_G^*(U)$ is the kernel of the composition

$$H^*(G, k) \xrightarrow{\delta_G} HH^*(kG) \xrightarrow{\alpha_U} \text{Ext}_{kG}^*(U, U),$$

where α_U is induced by the functor $- \otimes_{kG} U$. We denote by $V_{\mathcal{F}}$ the variety of the graded k -algebra $H^*(\mathcal{F})$. Let $I_{\mathcal{F}}^*(U)$ be the kernel of the composition of graded k -algebra homomorphisms

$$H^*(\mathcal{F}) \xrightarrow{\delta_P} HH^*(kP) \xrightarrow{\alpha_U} \text{Ext}_{kP}^*(U, U),$$

where in the right side U is regarded as kP -module by restriction. We define $V_{\mathcal{F}}(U)$ as the subvariety of $V_{\mathcal{F}}$ which consists of all maximal ideals containing $I_{\mathcal{F}}^*(U)$. For E a subgroup of P we denote by $r_{P,E}^* : V_E \rightarrow V_{\mathcal{F}}$ the map between varieties induced by the composition

$$H^*(\mathcal{F}) \xhookrightarrow{i_P} H^*(P, k) \xrightarrow{\text{res}_E^P} H^*(E, k).$$

We need to state the following remark which includes well-known results.

Remark 2.1. Let P be a p -subgroup of a finite group G such that $\mathcal{F} = \mathcal{F}_P(G)$ is a saturated fusion system on P . Let U be a finitely generated kG -module. Then $V_{\mathcal{F}}(U) = i_P^*(V_P(U))$ where $i_P^* : V_P \rightarrow V_{\mathcal{F}}$ is the induced finite surjective map. Moreover, by applying [7, Theorem 9.4.3] we have that $V_{\mathcal{F}}(U) = \bigcup_E r_{P,E}^*(V_E(U))$, where E runs over the set of elementary abelian subgroups of P .

For P a p -subgroup of a finite group G such that $\mathcal{F} = \mathcal{F}_P(G)$ is a saturated fusion system on P we denote by $\rho_{\mathcal{F},G} : H^*(G, k) \rightarrow H^*(\mathcal{F})$, the homomorphism of graded k -algebras induced by restriction.

Theorem 2.1. *Let P be a p -subgroup of a finite group G such that $\mathcal{F} = \mathcal{F}_P(G)$ is a saturated fusion system on P .*

- (i) *The homomorphism of k -algebras $\rho_{\mathcal{F},G}$ induces a finite map $\rho_{\mathcal{F},G}^* : V_{\mathcal{F}} \rightarrow V_G$.*
- (ii) *For any finitely generated kG -module U we have*

$$V_{\mathcal{F}}(U) = (\rho_{\mathcal{F},G}^*)^{-1}(V_G(U)).$$

Proof. Statement (i) is straightforward to check. Composing the commutative diagram from [9, Theorem 5.1], adapted to our situation, with the obvious commutative diagram (see [8, Proposition 4.7])

$$\begin{array}{ccc} H^*(G, k) & \xrightarrow{\delta_G} & HH_M^*(kG) , \\ \downarrow \rho_{\mathcal{F},G} & & \downarrow T_{M^*} \\ H^*(\mathcal{F}) & \xrightarrow{\delta_P} & HH^*(kP) \end{array}$$

we obtain the following commutative diagram

$$\begin{array}{ccc} H^*(G, k) & \xrightarrow{\alpha_U \circ \delta_G} & \text{Ext}_{kG}^*(U, U) , \\ \downarrow \rho_{\mathcal{F},G} & & \downarrow \beta_{M^*,U} \\ H^*(\mathcal{F}) & \xrightarrow{\alpha_{M^* \otimes_{kG} U} \circ \delta_P} & \text{Ext}_{kP}^*(U, U) \end{array}$$

where $\beta_{M^*,U}$ is defined by the functor $M^* \otimes_{kG} -$. It follows that $\rho_{\mathcal{F},G}(I_G^*(U)) \subseteq I_{\mathcal{F}}^*(U)$ (since $M^* \otimes_{kG} U = U$ as kP -module), and then $V_{\mathcal{F}}(U) \subseteq (\rho_{\mathcal{F},G}^*)^{-1}(V_G(U))$. Suppose now that $\alpha \in V_{\mathcal{F}}$ and $\beta \in V_G(U)$ such that $\beta = \rho_{\mathcal{F},G}^*(\alpha)$. We take $U = k$ as trivial kP -module in Remark 2.1 to obtain that there is an elementary abelian subgroup of P and $\alpha' \in V_E$ such

that $\alpha = r_{P,E}^*(\alpha')$. By [7, Theorem 9.4.3] there is $E' \leq P$ an elementary abelian subgroup and $\beta' \in V_{E'}(U)$ such that $\beta = (\text{res}_{E'}^G)^*(\beta')$. The following equalities hold

$$\begin{aligned} \rho_{\mathcal{F},G}^*(\alpha) &= \rho_{\mathcal{F},G}^*(r_{P,E}^*(\alpha')) = ((\text{res}_P^G)^* \circ (\text{res}_E^P \circ i_P)^*)(\alpha') = \\ &= (\text{res}_E^P \circ i_P \circ \text{res}_P^G)^*(\alpha') = (\text{res}_E^G)^*(\alpha'). \end{aligned}$$

We also have $\rho_{\mathcal{F},G}^*(\alpha) = \beta = (\text{res}_{E'}^G)^*(\beta')$. It follows that $(\text{res}_{E'}^G)^*(\beta') = (\text{res}_E^G)^*(\alpha')$. By [7, Proposition 9.6.1] we conclude that $\alpha' \in V_{E'}(U)$, hence $\alpha = r_{P,E}^*(\alpha') \in V_{\mathcal{F}}(U)$ (see Remark 2.1). \square

We end this section with the so called Subgroup Theorem ([7, Theorem 9.6.2]) for varieties associated to cohomology algebras of saturated fusion systems. Let \mathcal{F}_1 be a saturated fusion system on a p -group P_1 and \mathcal{F}_2 a saturated fusion system on a p -group P_2 such that \mathcal{F}_1 is a fusion subsystem of \mathcal{F}_2 , where P_1 is a subgroup of P_2 . It is easy to check that $\text{res}_{P_1}^{P_2}(\text{H}^*(\mathcal{F}_2)) \subseteq \text{H}^*(\mathcal{F}_1)$. We denote by

$$\text{res}_{\mathcal{F}_2, \mathcal{F}_1} : \text{H}^*(\mathcal{F}_2) \rightarrow \text{H}^*(\mathcal{F}_1)$$

the homomorphism of k -algebras defined by $\text{res}_{\mathcal{F}_2, \mathcal{F}_1}([\zeta]) = \text{res}_{P_1}^{P_2}([\zeta])$ for any $[\zeta] \in \text{H}^*(\mathcal{F}_2)$.

Theorem 2.2. *With the above assumptions and notations let U be a finitely generated kP_2 -module. Then*

$$V_{\mathcal{F}_1}(U) = (\text{res}_{\mathcal{F}_2, \mathcal{F}_1}^*)^{-1}(V_{\mathcal{F}_2}(U)).$$

Proof. Let X be kP_2 viewed as $kP_2 - kP_1$ -bimodule. It is easy to verify the commutativity of the following diagram

$$\begin{array}{ccc} \text{H}^*(\mathcal{F}_2) & \xrightarrow{\delta_{P_2}} & \text{HH}_X^*(kP_2) \\ \downarrow \text{res}_{\mathcal{F}_2, \mathcal{F}_1} & & \downarrow T_X^* \\ \text{H}^*(\mathcal{F}_1) & \xrightarrow{\delta_{P_1}} & \text{HH}_{X^*}^*(kP_1) \end{array}$$

We know that $I_{\mathcal{F}_1}(U) = \text{Ker}(\alpha_U \circ \delta_{P_1})$ and that $I_{\mathcal{F}_2}(U) = \text{Ker}(\alpha_U \circ \delta_{P_2})$. By [9, Theorem 5.1] the following diagram is commutative

$$\begin{array}{ccc} \text{HH}_X^*(kP_2) & \xrightarrow{\alpha_U} & \text{Ext}_{kP_2}^*(U, U) \\ \downarrow T_X^* & & \downarrow \beta_{X^*, U} \\ \text{HH}_{X^*}^*(kP_1) & \xrightarrow{\alpha_{X^* \otimes_{kP_1} U}} & \text{Ext}_{kP_1}^*(U, U) \end{array}$$

The above diagrams assure us the inclusion $\text{res}_{\mathcal{F}_2, \mathcal{F}_1}^*(V_{\mathcal{F}_1}(U)) \subseteq V_{\mathcal{F}_2}(U)$. For the reverse inclusion we use Remark 2.1 and [7, Proposition 9.6.1]. \square

Remark 2.2. The above theorems could have applications in studying when $\rho_{\mathcal{F}, G}$ and $\text{res}_{\mathcal{F}_2, \mathcal{F}_1}$ induce bijective maps on varieties, or when are F -isomorphisms. A missing ingredient for doing this is the Quillen stratification for fusion systems which is not appeared yet in a suitable form in the literature. We do not pursue this in here. However, for saturated fusion systems associated to blocks we obtain some positive answers; as we will see in Section 3.

3. A theorem of Alperin for cohomology of fusion systems associated to blocks

In this section we work only with cohomology algebras of saturated fusion systems associated to p -blocks. Let H be a finite subgroup of G . Let b be a block of kG with defect pointed group P_γ and let c be a block of kH with defect group Q_δ such that $Q \leq P$. Let $i \in \gamma$ be a source idempotent. By Broué and Puig [5, Theorem 1.8] for any subgroup R of P , there is a unique block e_R of $C_G(R)$ such that $\text{Br}_R(i)e_R \neq 0$. Then (R, e_R) is a b -Brauer pair, and e_R is also the unique block of $C_G(R)$ such that $(R, e_R) \leq (P, e_P)$. We define the saturated fusion system $\mathcal{F}_{(P, e_P)}(G, b)$ the category which has as objects the set of subgroups of P ; for any two subgroups R, S of P the set of morphisms from R to S in $\mathcal{F}_{(P, e_P)}(G, b)$ is the set of (necessarily injective) group homomorphisms $\varphi : R \rightarrow S$ for which there is an element $x \in G$ satisfying $\varphi(u) = xux^{-1}$ for all $u \in R$ and satisfying ${}^x(R, e_R) \leq (S, e_S)$. For shortness we will use the notation $\mathcal{F}_{G, b}$ for this category. The category $\mathcal{F}_{G, b}$ is, up to canonical isomorphism of categories, independent of the choice of the source idempotent i . The analogous definition holds for $\mathcal{F}_{H, c}$; here we denote by (R, f_R) an c -Brauer pair, for $R \leq Q$. We denote by $\mathcal{E}_P(b)$, (respectively $\mathcal{E}_Q(c)$) the set of all elementary abelian subgroups of P , (respectively Q). We mention that the cohomology algebra of the block b , which is $H^*(\mathcal{F}_{G, b})$, was introduced and analyzed before the cohomology of general fusion systems by M. Linckelmann in some papers ([8], [9], [10]) around the beginning of 2000's.

For the rest of the section let b, c be blocks as above such that $\mathcal{F}_{H, c}$ is a fusion subsystem of $\mathcal{F}_{G, b}$. It is easy to check that res_Q^P induces a well-defined homomorphism of algebras

$$\text{res}_{b, c} : H^*(\mathcal{F}_{G, b}) \rightarrow H^*(\mathcal{F}_{H, c}).$$

Obviously $\text{res}_{b, c}$ is the map $\text{res}_{\mathcal{F}_{G, b}, \mathcal{F}_{H, c}}$ defined at the end of Section 2.

In the following lines we introduce some notations adapted to our setting, regarding the Quillen stratification of $H^*(\mathcal{F}_{G, b})$ obtained in [10]. Let $E \in \mathcal{E}_P(b)$. The restriction map $r_{P, E} : H^*(\mathcal{F}_{G, b}) \rightarrow H^*(E, k)$ induces a map on varieties, which we denote $r_{E, b}^* : V_E \rightarrow V_{\mathcal{F}_{G, b}}$.

As usual we define the subvariety of V_E

$$V_E^+ = V_E \setminus \bigcup_{F < E} (\text{res}_F^E)^*(V_F),$$

and denote the subvarieties of $V_{\mathcal{F}_{G,b}}$

$$V_{G,E,b} = r_{E,b}^*(V_E), \quad V_{G,E,b}^+ = r_{E,b}^*(V_E^+).$$

Finally we set $W_G(E) = N_G(E, e_E)/EC_G(E)$, where (E, e_E) is the unique b -Brauer pair such that $(E, e_E) \leq (P, e_P)$. If $E \in \mathcal{E}_Q(c)$ we consider the analogous notations:

$$r_{E,c}^*, V_{H,E,c}, V_{H,E,c}^+, W_H(E) = N_G(E, f_E)/EC_H(E).$$

Now the Quillen stratification is

$$V_{\mathcal{F}_{G,b}} = \bigcup_E^\circ V_{G,E,b}^+,$$

where E runs over a set of $\mathcal{F}_{G,b}$ -conjugacy classes of elements in $\mathcal{E}_P(b)$.

Definition 3.1. Let b, c be blocks of kG , respectively kH such that $\mathcal{F}_{H,c}$ is a fusion subsystem of $\mathcal{F}_{G,b}$. We say that $\mathcal{F}_{H,c}$ is *weakly elementary embedded* in $\mathcal{F}_{G,b}$ if the following conditions are satisfied:

- (1) Whenever $E \in \mathcal{E}_Q(c)$ then $W_H(E) \cong W_G(E)$;
- (2) Every subgroup from $\mathcal{E}_P(b)$ is $\mathcal{F}_{G,b}$ -conjugate to a subgroup from $\mathcal{E}_Q(c)$;
- (3) If two subgroups from $\mathcal{E}_Q(c)$ are $\mathcal{F}_{G,b}$ -conjugate then they are $\mathcal{F}_{H,c}$ -conjugate.

Theorem 3.1. Let b, c be blocks of kG , respectively kH such that $\mathcal{F}_{H,c}$ is a fusion subsystem of $\mathcal{F}_{G,b}$. The restriction map $\text{res}_{b,c} : H^*(\mathcal{F}_{G,b}) \rightarrow H^*(\mathcal{F}_{H,c})$ induces a bijective map $\text{res}_{b,c}^* : V_{\mathcal{F}_{H,c}} \rightarrow V_{\mathcal{F}_{G,b}}$ if and only if $\mathcal{F}_{H,c}$ is weakly elementary embedded in $\mathcal{F}_{G,b}$.

Proof. Suppose that the conditions from Definition 3.1 are satisfied. For injectivity let $m_1, m_2 \in V_{\mathcal{F}_{H,c}}$ such that $\text{res}_{b,c}^*(m_1) = \text{res}_{b,c}^*(m_2)$. By the Quillen stratification for $V_{\mathcal{F}_{H,c}}$ there are two subgroups $E_1, E_2 \in \mathcal{E}_Q(c)$ unique up to $\mathcal{F}_{H,c}$ -conjugacy, and $\gamma_1 \in V_{E_1}^+, \gamma_2 \in V_{E_2}^+$ such that

$$m_1 = r_{E_1,c}^*(\gamma_1), \quad m_2 = r_{E_2,c}^*(\gamma_2).$$

It follows that

$$(\text{res}_{b,c}^* \circ r_{E_1,c}^*)(\gamma_1) = (\text{res}_{b,c}^* \circ r_{E_2,c}^*)(\gamma_2),$$

and since

$$r_{E_1,c} \circ \text{res}_{b,c} = r_{E_1,b}, \quad r_{E_2,c} \circ \text{res}_{b,c} = r_{E_2,b},$$

we obtain that $r_{E_1,b}^*(\gamma_1) = r_{E_2,b}^*(\gamma_2)$. Quillen stratification for $V_{\mathcal{F}_{G,b}}$ give us that E_1 is $\mathcal{F}_{G,b}$ -conjugate to E_2 , hence E_1 is $\mathcal{F}_{H,c}$ -conjugate to E_2 (by condition (3) of Definition 3.1). This allows us to choose $E_1 = E_2 = E$, and we have $r_{E,b}^*(\gamma_1) = r_{E,b}^*(\gamma_2)$. The inseparable isogeny from [10, Theorem 4.2, Proposition 4.3], $V_{G,E,b}^+ \cong V_E^+ / W_G(E)$, is given by $r_{E,b}^*$, hence γ_1, γ_2 are in the same orbit of the action of $W_G(E)$ on V_E^+ . From Definition 3.1, condition (1) we obtain that γ_1, γ_2 are in the same orbit of the action of $W_H(E)$ on V_E^+ , hence $m_1 = r_{E,c}^*(\gamma_1) = r_{E,c}^*(\gamma_2) = m_2$. Now let $m \in V_{\mathcal{F}_{G,b}}$. There is $F \in \mathcal{E}_P(b)$ and $\gamma \in V_F^+$ such that $m = r_{F,b}^*(\gamma)$. From condition (2) of the above definition we know that there is $F' \in \mathcal{E}_Q(c)$ such that F is $\mathcal{F}_{G,b}$ -conjugate to F' . Then $r_{F,b}^*(\gamma) = r_{F',b}^*(\gamma)$ and $m = r_{F',b}^*(\gamma) = \text{res}_{b,c}^*(r_{F',c}^*(\gamma))$.

Now we assume that $\text{res}_{b,c}^*$ is bijective. Let $E_1, E_2 \in \mathcal{E}_Q(c)$ which are $\mathcal{F}_{G,b}$ -conjugate. Then $V_{G,E_1,b}^+ = V_{G,E_2,b}^+$, that is $r_{E_1,b}^*(V_{E_1}^+) = r_{E_2,b}^*(V_{E_2}^+)$. It follows that

$$\text{res}_{b,c}^*(r_{E_1,c}^*(V_{E_1}^+)) = \text{res}_{b,c}^*(r_{E_2,c}^*(V_{E_2}^+)).$$

Since $\text{res}_{b,c}^*$ is injective we obtain that

$$V_{H,E_1,c}^+ = r_{E_1,c}^*(V_{E_1}^+) = r_{E_2,c}^*(V_{E_2}^+) = V_{H,E_2,c}^+,$$

and then E_1, E_2 are $\mathcal{F}_{H,c}$ -conjugate, hence (3) is proved. For (1) let $E \in \mathcal{E}_Q(c)$. We identify $V_{G,E,b}^+$ with the orbits of the action of $W_G(E)$ on V_E^+ and the same for $V_{H,E,c}^+$. Since $\text{res}_{b,c}^*$ is bijective and

$$\text{res}_{b,c}^*(V_{H,E,c}^+) = \text{res}_{b,c}^*(r_{E,c}^*(V_E^+)) = r_{E,b}^*(V_E^+) = V_{G,E,b}^+,$$

we obtain that $\text{res}_{b,c}^*$ induces a bijection from $V_{H,E,c}^+$ to $V_{G,E,b}^+$. This bijection, the free actions of $W_G(E), W_H(E)$ on V_E^+ and the existence of an embedding from $W_H(E)$ into $W_G(E)$ assure the validity of condition (1); how $\text{res}_{b,c}^*$ is defined is an important ingredient for proving isomorphism (1). For (2) we consider $n \in V_{\mathcal{F}_{H,c}}$ and $m \in V_{\mathcal{F}_{G,b}}$ with vertex $E \in \mathcal{E}_P(b)$ and a source $\gamma \in V_E^+$ such that $\text{res}_{b,c}^*(n) = m$. We get that a $\mathcal{F}_{G,b}$ -conjugate of E must be a vertex of n , hence in $\mathcal{E}_Q(c)$. \square

Remark 3.2. With the hypothesis from Theorem 3.1 if b, c are the principal blocks we obtain an analogous statement to [1, Theorem 2].

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